

# SOME RESULTS ON THE SCHWARTZ SPACE OF $\Gamma \backslash G$

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**ABSTRACT.** Let  $G$  be a connected semisimple Lie group with finite center. Let  $\Gamma \subset G$  be a discrete subgroup. We study closed admissible irreducible subrepresentations of the space of distributions  $\mathcal{S}(\Gamma \backslash G)'$  defined by Casselman [9], and their relations to automorphic forms.

## 1. INTRODUCTION

Let  $G$  be a connected semisimple Lie group with finite center. Let  $K$  be the maximal compact subgroup of  $G$ , and  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$  the center of the universal enveloping algebra of the complexification of the Lie algebra  $\mathfrak{g}$  of  $G$ . Let  $\Gamma \subset G$  be a discrete subgroup. For example, it could be a trivial group. But the main example is given by the following

**Assumptions 1-1.** *We assume that  $G$  is a group of  $\mathbb{R}$ -points of a semisimple algebraic group  $\mathcal{G}$  defined over  $\mathbb{Q}$ . Assume that  $G$  is not compact and connected. Let  $\Gamma \subset G$  be congruence subgroup with respect to the arithmetic structure given by the fact that  $\mathcal{G}$  defined over  $\mathbb{Q}$  (see [7]).*

In [9], Casselman has defined the Schwartz space  $\mathcal{S}(\Gamma \backslash G)$  (see Section 3 for definition). It is obvious that  $G$  acts on the right. The corresponding representation is a smooth representation of moderate growth ([8], [25]). The main object of the interest is the strong topological dual space  $\mathcal{S}(\Gamma \backslash G)'$ . This is the space of all continuous linear functionals on  $\mathcal{S}(\Gamma \backslash G)$  equipped with the strong topology. By general theory of topological vector spaces, the space  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex vector space. The natural action of  $G$  on  $\mathcal{S}(\Gamma \backslash G)'$  is continuous. The usual representation-theoretic arguments are valid there ([13], Section 2).

The main interest in the space  $\mathcal{S}(\Gamma \backslash G)'$  is that its Garding space can be identified with the space of functions of uniform moderate growth  $\mathcal{A}_{umg}(\Gamma \backslash G)$  (see (UMG-1) and (UMG-2) in Section 3 for the definition). Under Assumption 1-1,  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite  $\mathcal{A}_{umg}(\Gamma \backslash G)$  are smooth automorphic forms on  $G$  for  $\Gamma$ . Also,  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite and  $K$ -finite on the right in  $\mathcal{A}_{umg}(\Gamma \backslash G)$  are equal to the space usual space  $\mathcal{A}(\Gamma \backslash G)$  of  $K$ -finite automorphic forms for  $\Gamma$  [7].

Now, we describe the content of the paper and main results proved in the paper. In Section 2, under Assumption 1-1, we recall the notion of smooth and  $K$ -finite automorphic forms. In Section 3, we describe the results of Casselman [9] used in the paper. In Section 4 we prove some main results in the paper. This section is strongly motivated by a lecture of Wallach [26]. Some of the results here are probably well-known, and we present our way of

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understanding them. We let  $(\pi, \mathcal{H})$  be an irreducible admissible representation of  $G$  acting on the Hilbert space  $\mathcal{H}$ . The space of  $\mathcal{H}^\infty$  vectors in  $\mathcal{H}$  is a representation of moderate growth. The main results of Section 4 gives the description of closed irreducible admissible subrepresentations of  $\mathcal{S}(\Gamma \backslash G)'$  in terms of continuous  $\Gamma$ -invariant functionals on  $\mathcal{H}^\infty$  (see Proposition 4-4, Theorem 4-9). The proofs use deep results of Casselman and Wallach ([8], [25]) on smooth globalization of representations at the critical points. Examples of subrepresentation can explicitly be constructed using Eisenstein series [14], or be shown to exists using Poincaré series ([18], [19], [20], [21]), or the trace formula ([1], [2]). In Theorem 4-10, we prove that the trivial representation is the only finite-dimensional subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$  under Assumption 1-1 and assuming that  $G$  has no compact components. In Section 5, we study realization inside  $\mathcal{S}(\Gamma \backslash G)'$  of irreducible subrepresentations  $\mathcal{H}$  of  $L^2(\Gamma \backslash G)$  (see Theorem 5-8). In this case,  $\mathcal{H}^\infty \subset \mathcal{A}_{umg}(\Gamma \backslash G)$ . The proof of Theorem 5-8 contains the proof of the fact that smooth cuspidal automorphic forms are rapidly decreasing. This is proved using methods of Casselman and Wallach. Different proof is contained in [17]. In Theorem 4-8, we relate various topologies on  $\mathcal{H}^\infty$  for an irreducible subspace  $\mathcal{H} \subset L^2(\Gamma \backslash G)$ . For example, we prove that if the sequence of elements in  $\mathcal{H}^\infty$ ,  $(\varphi_n)_{n \geq 1}$ , converges to  $\varphi \in \mathcal{H}^\infty$  in the standard topology on  $\mathcal{H}^\infty$ , then it converges to  $\varphi$  in usual topology on  $C^\infty(G)$  (see the description before the statement of Theorem 5-9). In Section 6, we study  $\Gamma$ -invariants in  $\mathcal{S}'(G)$  and their relation to the space  $(\mathcal{S}(G)')^\Gamma$  (see Proposition 6-5). In Proposition 6-6 we give the interpretaion of the classical construction of automorphic via Poincaré series (see for example [21]) in terms of  $\Gamma$ -invariants in  $\mathcal{S}'(G)$ .

## 2. PRELIMINARIES

In this section we assume that  $G$  is a connected semisimple Lie group with finite center, and recall the notion of the norm on  $G$ . It is essential for all what follows.

We fix a minimal parabolic subgroup  $P = MAN$  of  $G$  in the usual way (see [24], Section 2). We have the Iwasawa decomposition  $G = NAK$ .

We recall the notion of a norm on the group following [24], 2.A.2. A norm  $\| \cdot \|$  is a continuous function  $G \rightarrow [1, \infty[$  satisfying the following properties:

- (1)  $\|x^{-1}\| = \|x\|$ , for all  $x \in G$ ;
- (2)  $\|x \cdot y\| \leq \|x\| \cdot \|y\|$ , for all  $x, y \in G$ ;
- (3) the sets  $\{x \in G; \|x\| \leq r\}$  are compact for all  $r \geq 1$ ;
- (4)  $\|k_1 \exp(tX)k_2\| = \|\exp(X)\|^t$ , for all  $k_1, k_2 \in K, X \in \mathfrak{p}, t \geq 0$ .

Any two norms  $\| \cdot \|_i, i = 1, 2$ , are equivalent: there exist  $C, r > 0$  such that  $\|x\|_1 \leq C\|x\|_2^r$ , for all  $x \in G$ .

We recall the following lemma:

**Lemma 2-1.** *There exists a real number  $d_0 > 0$  such that  $\int_G \|g\|^{-d} dg < \infty$  for  $d \geq d_0$ . Since  $\|g\| \geq 1$  for all  $g \in G$ , the lemma follows.*

*Proof.* The existence of  $d_0 > 0$  such that  $\int_G \|g\|^{-d_0} dg < \infty$  is proved in ([24], Lemma 2.A.2.4).  $\square$

In the remainder of this section, we assume the following:

**Assumptions 2-2.** *We assume that  $G$  is a group of  $\mathbb{R}$ -points of a semisimple algebraic group  $\mathcal{G}$  defined over  $\mathbb{Q}$ . Assume that  $G$  is not compact and connected. Let  $\Gamma \subset G$  be congruence subgroup with respect to the arithmetic structure given by the fact that  $\mathcal{G}$  defined over  $\mathbb{Q}$  (see [7]).*

The group satisfying the Assumption 2-2 is a connected semisimple Lie group with finite center. Also,  $\Gamma$  is a discrete subgroup of  $G$  and it has a finite covolume.

An automorphic form (or a  $K$ -finite automorphic form; see [11]) for  $\Gamma$  is a function  $f \in C^\infty(G)$  satisfying the following three conditions ([26] or [7]):

- (A-1)  $f$  is  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite and  $K$ -finite on the right;
- (A-2)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma$ ,  $x \in G$ ;
- (A-3) there exists  $r \in \mathbb{R}$ ,  $r > 0$  such that for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|^r$ , for all  $x \in G$ .

A smooth automorphic form (see [9], [11]) for  $\Gamma$  is a function  $f \in C^\infty(G)$  satisfying (A1)–(A3) except possibly  $K$ -finiteness. We discuss smooth automorphic forms in more detail the next section.

We write  $\mathcal{A}(\Gamma \backslash G)$  (resp.,  $\mathcal{A}^\infty(\Gamma \backslash G)$ ) for the vector space of all automorphic forms (resp., smooth automorphic forms). Obviously,  $\mathcal{A}(\Gamma \backslash G) \subset \mathcal{A}^\infty(\Gamma \backslash G)$ . It is easy to see that  $\mathcal{A}(\Gamma \backslash G)$  is a  $(\mathfrak{g}, K)$ -module (using [13], Theorem 1), and since  $G$  is connected, the space  $\mathcal{A}^\infty(\Gamma \backslash G)$  is  $G$ -invariant. An automorphic form  $f \in \mathcal{A}^\infty(\Gamma \backslash G)$  is a  $\Gamma$ -cuspidal automorphic form if for every proper  $\mathbb{Q}$ -proper parabolic subgroup  $\mathcal{P} \subset \mathcal{G}$  we have

$$\int_{U \cap \Gamma \backslash U} f(ux) dx = 0, \quad x \in G,$$

where  $U$  is the group of  $\mathbb{R}$ -points of the unipotent radical of  $\mathcal{P}$ . We remark that the quotient  $U \cap \Gamma \backslash U$  is compact. We use normalized  $U$ -invariant measure on  $U \cap \Gamma \backslash U$ . The space of all  $\Gamma$ -cuspidal automorphic forms (resp.,  $\Gamma$ -cuspidal smooth automorphic forms) for  $\Gamma$  is denoted by  $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$  (resp.,  $\mathcal{A}_{\text{cusp}}^\infty(\Gamma \backslash G)$ ). The space  $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$  is a  $(\mathfrak{g}, K)$ -submodule of  $\mathcal{A}(\Gamma \backslash G)$ . The space  $\mathcal{A}_{\text{cusp}}^\infty(\Gamma \backslash G)$  is  $G$ -invariant.

Following Casselman [9], we define

$$\|g\|_{\Gamma \backslash G} = \inf_{\gamma \in \Gamma} \|\gamma g\|, \quad g \in G.$$

It is obvious that  $\|\cdot\|_{\Gamma \backslash G}$  is  $\Gamma$ -invariant on the right, and that  $\|g\|_{\Gamma \backslash G} \leq \|g\|$  for all  $g \in G$ . The condition (A-3) is equivalent to

- (A-3') there exists  $r \in \mathbb{R}$ ,  $r > 0$  such that for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|_{\Gamma \backslash G}^r$ , for all  $x \in G$ .

We recall the following standard result:

**Lemma 2-3.** *Under above assumptions, we have the following:*

- (a) *If  $f \in C^\infty(G)$  satisfies (A-1), (A-2), and there exists  $p \geq 1$  such that  $f \in L^p(\Gamma \backslash G)$ , then  $f$  satisfies (A-3), and it is therefore an automorphic form. We speak about  $p$ -integrable automorphic form, for  $p = 1$  (resp.,  $p = 2$ ) we speak about integrable (resp., square-integrable) automorphic form.*
- (b) *Let  $p \geq 1$ . Every  $p$ -integrable automorphic form is integrable.*
- (c) *Bounded integrable automorphic form is square-integrable.*
- (d) *If  $f$  is square integrable automorphic form, then the minimal  $G$ -invariant closed subspace of  $L^2(\Gamma \backslash G)$  is a direct sum of finitely many irreducible unitary representations.*
- (e) *Every  $\Gamma$ -cuspidal automorphic form is square-integrable.*

*Proof.* For the claims (a) and (e) we refer to [7] and reference there. Since the volume of  $\Gamma \backslash G$  is finite, the claim (b) follows from Hölder inequality (as in [18], Section 3). The claim (c) is obvious. The claim (d) follows from ([24], Corollary 3.4.7 and Theorem 4.2.1).  $\square$

In ([21], Proposition 4.7) we give a simple proof of Lemma 2-3 (a) using results of Casselman [9] recalled in the next section.

### 3. SOME RESULTS OF CASSELMAN

In this section we assume that  $G$  is a semisimple connected Lie group with finite center. We assume that  $\Gamma$  is a discrete subgroup of  $G$ . For example,  $\Gamma$  could be a congruence subgroup or just a trivial group. We recall the definition of the Schwartz space  $\mathcal{S}(\Gamma \backslash G)$

defined by Casselman ([9], page 292). It consists of all functions  $f \in C^\infty(G)$  satisfying the following conditions:

- (CS-1)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma$ ,  $x \in G$ ;
- (CS-2)  $\|f\|_{u, -n} < \infty$  for all  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , and all natural numbers  $n \geq 1$ .

In above definition, for  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , and a real number  $s$ , we let

$$\|f\|_{u, s} \stackrel{\text{def}}{=} \sup_{x \in G} \|x\|_{\Gamma \backslash G}^{-s} |u.f(x)|.$$

Since  $\|x\|_{\Gamma \backslash G} \geq 1$ , we have

$$\|f\|_{u, s'} \leq \|f\|_{u, s},$$

for  $s' > s$ .

We recall the following result (see [9], 1.8 Proposition):

**Proposition 3-1.** *Using above notation, we have the following:*

- (i) *The Schwartz space  $\mathcal{S}(\Gamma \backslash G)$  is a Fréchet space under the seminorms:  $\|\cdot\|_{u, -n}$ ,  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ ,  $n \in \mathbb{Z}_{\geq 1}$ .*
- (ii) *The right regular representation of  $G$  on  $\mathcal{S}(\Gamma \backslash G)$  is a smooth Fréchet representation of moderate growth.*

We recall the definition of representation of moderate growth. Let  $(\pi, V)$  be a continuous representation on the Fréchet space  $V$ . We say that  $(\pi, V)$  is of moderate growth if it is smooth and if for any continuous semi-norm  $\rho$  there exists an integer  $n$ , a constant  $C > 0$ , and another continuous semi-norm  $\nu$  such that

$$\|\pi(g)v\|_\rho \leq C\|g\|^n\|v\|_\nu, \quad g \in G, \quad v \in V.$$

We recall that the semi-norms on a locally convex vector space (for example, a Fréchet space)  $V$  are constructed via Minkowski functionals.

The following definition is from ([9], page 295).

**Definition 3-2.** *The space  $\mathcal{S}(\Gamma \backslash G)'$  of tempered distributions or distributions of moderate growth on  $\Gamma \backslash G$  is the strong topological dual of  $\mathcal{S}(\Gamma \backslash G)$ .*

For convenience of the reader, we recall the definition of a strong topological dual in our particular case. By general theory, the subset  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded if for every neighborhood  $V$  of 0 there exists  $s > 0$  such that  $B \subset tV$ , for  $t > s$ . This definition is not very practical to use. Again from the general theory (and easy to see directly),  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded if and only if it is bounded in every semi-norm defining topology on  $\mathcal{S}(\Gamma \backslash G)$  i.e.,

$$\sup_{f \in B} \|f\|_{u, -n} < \infty, \quad u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}), \quad n \in \mathbb{Z}_{\geq 1}.$$

The strong topological dual  $\mathcal{S}(\Gamma \backslash G)'$  of  $\mathcal{S}(\Gamma \backslash G)$  is the space of continuous functionals on  $X$  equipped with strong topology i.e. topology of uniform convergence on bounded sets in  $\mathcal{S}(\Gamma \backslash G)$  i.e. topology given by semi-norms

$$\|\alpha\|_B = \sup_{f \in B} |\alpha(f)|, \quad \text{where } B \text{ ranges over bounded sets of } \mathcal{S}(\Gamma \backslash G).$$

By general theory of topological vector spaces, the space  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex (defined by above semi-norms) vector space.

The natural action of  $G$  on  $\mathcal{S}(\Gamma \backslash G)'$  is continuous. The usual representation-theoretic arguments are valid there ([13], Section 2).

Following Casselman, we consider the two spaces of functions: the functions of moderate growth  $\mathcal{A}_{mg}(\Gamma \backslash G)$ , and the functions of uniform moderate growth  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . The space  $\mathcal{A}_{mg}(\Gamma \backslash G)$  consists of the functions  $f \in C^\infty(G)$  satisfying the following conditions:

- (MG-1)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma, x \in G$ ;
- (MG-2) for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0, r_u \in \mathbb{R}, r_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|^{r_u}$ , for all  $x \in G$ .

The space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  consists of the functions  $f \in C^\infty(G)$  satisfying the following conditions:

- (UMG-1)  $f$  is left-invariant under  $\Gamma$  i.e.,  $f(\gamma x) = f(x)$  for all  $\gamma \in \Gamma, x \in G$ ;

(UMG-2) there exists  $r \in \mathbb{R}$ ,  $r > 0$  such that for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  there exists a constant  $C_u > 0$  such that  $|u.f(x)| \leq C_u \cdot \|x\|^r$ , for all  $x \in G$ .

We note that in the second definition  $r$  is independent of  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ .

**Lemma 3-3.** *We maintain the assumptions of the first paragraph of Section 2. Then, the spaces of functions which are  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite and  $K$ -finite on the right in  $\mathcal{A}_{mg}(\Gamma \backslash G)$ , and in  $\mathcal{A}_{umg}(\Gamma \backslash G)$  coincide, and are equal to the space  $\mathcal{A}(\Gamma \backslash G)$  of automorphic forms for  $\Gamma$ . Next, the space of smooth automorphic forms  $\mathcal{A}^{\infty}(\Gamma \backslash G)$  is a subspace of  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite functions in  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . Furthermore, we have*

$$\mathcal{A}(\Gamma \backslash G) \subset \mathcal{A}^{\infty}(\Gamma \backslash G) \subset \mathcal{A}_{umg}(\Gamma \backslash G) \subset \mathcal{A}_{mg}(\Gamma \backslash G).$$

*Proof.* This is a simple observation made in ([21], Lemma 4.4). □

**Lemma 3-4.** *The Garding space in  $\mathcal{S}(\Gamma \backslash G)'$  is equal to the space  $\mathcal{A}_{umg}(\Gamma \backslash G)$ .*

*Proof.* This ([9], Theorem 1.16). □

We remark that  $\mathcal{S}(\Gamma \backslash G)'$  is not a Fréchet space so [12] can not be applied to prove that the space of smooth vectors is the same as the Garding space. Therefore, for example, in the settings of Lemma 3-3,  $\mathcal{A}^{\infty}(\Gamma \backslash G)$  is just subspace of the space of all  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite vectors in  $\mathcal{S}(\Gamma \backslash G)'$ .

Regarding smooth vectors in  $\mathcal{S}(\Gamma \backslash G)'$ , the following lemma will be used later (see [21], Lemma 4.6):

**Lemma 3-5.** *Assume that  $f \in L^p(\Gamma \backslash G)$ , for some  $p \geq 1$ , and  $\alpha \in C_c^{\infty}(G)$ . Then,  $f \star \alpha$  is equal almost everywhere to a function in  $\mathcal{A}_{umg}(\Gamma \backslash G)$ .*

#### 4. SOME RESULTS ON THE SPACES $\mathcal{S}(\Gamma \backslash G)'$

This section is strongly motivated by a lecture of Wallach [26]. Some of the results here are probably well-known, and we present our way of understanding them. We also give a complete description of irreducible closed subrepresentations  $\mathcal{S}(\Gamma \backslash G)'$ . We prove that under proper assumptions on  $G$  and  $\Gamma$  only finite dimensional subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$  is trivial representation.

In this section, we let  $(\pi, \mathcal{H})$  be an irreducible admissible representation of  $G$  acting on the Hilbert space  $\mathcal{H}$ . We write  $\langle \cdot, \cdot \rangle$  for the inner product on  $\mathcal{H}$ . We denote by  $\mathcal{H}^{\infty}$  the subspace of smooth vectors in  $\mathcal{H}$ . It is a complete Fréchet space under the family of semi-norms:

$$\|h\|_u = \|\pi(u)h\|, \quad u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}),$$

where  $\|\cdot\|$  is the norm on  $\mathcal{H}$  derived from  $\langle \cdot, \cdot \rangle$ . It is a smooth Frechét representation of moderate growth ([25], Lemma 11.5.1). In particular, if  $\lambda$  is a continuous functional on  $\mathcal{H}^{\infty}$ ,

then there exists  $d \in \mathbb{R}$ , and a continuous semi-norm  $\kappa$  such that

$$(4-1) \quad |\lambda(\pi(g)h)| \leq \|g\|^d \kappa(h), \quad g \in G, \quad h \in \mathcal{H}^\infty.$$

The reader can easily check that if (4-1) holds for any  $d = d_0$ , then it holds for all  $d \geq d_0$ . We make the following definition (see also [20], (3-4)):

**Definition 4-2.** Let  $d_{\mathcal{H}, \lambda} = d_{\pi, \lambda} \geq -\infty$  be the infimum of all  $d \in \mathbb{R}$  such that (4-1) holds for some continuous semi-norm  $\kappa = \kappa_d$ .

**Lemma 4-3.** The Frechét representation  $G$  on  $\mathcal{H}^\infty$  is irreducible in the category of Frechét representations.

*Proof.* This representation is a canonical globalization (see [25], Chapter 11, or [8]) of a  $(\mathfrak{g}, K)$ -module  $\mathcal{H}_K$ . Hence, the lemma. It is also to give a direct proof. Let  $\mathcal{V} \subset \mathcal{H}^\infty$  be a closed subrepresentation different than  $\{0\}$ . Pick any  $v \in \mathcal{V}$ ,  $v \neq 0$ . Then since  $\mathcal{H}^\infty$  is a smooth representation, the Fourier expansion converges absolutely ([13], Lemma 5):

$$v = \sum_{\delta \in \hat{K}} E_\delta(v),$$

where we fix the normalized Haar measure  $dk$  on  $K$ , and let

$$E_\delta(v) = \int_K d(\delta) \overline{\xi_\delta(k)} \pi(k)v \, dk.$$

Here, as usual  $\hat{K}$  is the set of equivalence of irreducible representations of  $K$ . Also, for  $\delta \in \hat{K}$ , we write  $d(\delta)$  and  $\xi_\delta$  for the degree and character of  $\delta$ , respectively. The vector  $E_\delta(v)$  belongs to the  $\delta$ -isotypic component  $\mathcal{V}(\delta)$  of  $\mathcal{V}$ . This shows that  $\mathcal{H}_K \cap \mathcal{V}$  is dense in  $\mathcal{V}$ . In particular,  $\mathcal{H}_K \cap \mathcal{V}$  is non-zero  $(\mathfrak{g}, K)$ -submodule of  $\mathcal{H}_K$ . Hence,  $\mathcal{H}_K \subset \mathcal{V}$  since  $\mathcal{H}_K$  is irreducible. But because of the same reason  $\mathcal{H}_K$  is dense in  $\mathcal{H}^\infty$ . This implies that  $\mathcal{V} = \mathcal{H}^\infty$ .  $\square$

**Proposition 4-4.** Let  $\Gamma \subset G$  be a discrete subgroup. Let  $\lambda$  be a continuous functional on  $\mathcal{H}^\infty$  which is  $\Gamma$ -invariant. Then, we have the following:

- (i) The pairing  $\mathcal{H}^\infty \times \mathcal{S}(\Gamma \backslash G) \rightarrow \mathbb{C}$  given by  $(h, f) \mapsto \int_{\Gamma \backslash G} \lambda(\pi(g)h) f(g) dg$  is well-defined, continuous, and  $G$ -equivariant.
- (ii) The map  $\mathcal{H}^\infty \rightarrow \mathcal{S}(\Gamma \backslash G)'$  which maps  $h \mapsto \alpha_{\lambda, \Gamma}(h)$  where

$$\alpha_{\lambda, \Gamma}(h)(f) = \int_{\Gamma \backslash G} \lambda(\pi(g)h) f(g) dg, \quad f \in \mathcal{S}(\Gamma \backslash G),$$

is a continuous map of locally convex representations of  $G$ . The image is contained in  $\mathcal{A}_{umg}(\Gamma \backslash G)$ .

- (iii) If  $\lambda \neq 0$ , then  $\alpha_{\lambda, \Gamma}$  is an embedding. The closure  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is a closed irreducible admissible subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$ .

*Proof.* We prove (i). First, we may assume that  $d > 0$  in (4-1). Then,  $\Gamma$ -invariance implies that

$$|\lambda(\pi(g)h)| = |\lambda(\pi(\gamma g)h)| \leq \|\gamma g\|^d \kappa(h),$$

for all  $\gamma \in \Gamma$ ,  $g \in G$ , and  $h \in \mathcal{H}^\infty$ . Hence

$$|\lambda(\pi(g)h)| \leq \|g\|_{\Gamma \backslash G}^d \kappa(h),$$

$g \in G$ , and  $h \in \mathcal{H}^\infty$ .

Next,  $\int_G \|g\|^{-d_0} dg < \infty$  for all sufficiently large  $d_0 > 0$ . Then, ([9], Proposition 1.9) implies that  $\int_{\Gamma \backslash G} \|g\|_{\Gamma \backslash G}^{-d_0} dg < \infty$  for all sufficiently large  $d_0 > 0$ . Hence

$$(4-5) \quad \left| \int_{\Gamma \backslash G} \lambda(\pi(g)h) f(g) dg \right| \leq \int_{\Gamma \backslash G} |\lambda(\pi(g)h) f(g)| dg \leq \kappa(h) \|f\|_{1, -d_0} \cdot \int_{\Gamma \backslash G} \frac{1}{\|g\|_{\Gamma \backslash G}^{-d+d_0}} dg.$$

Consequently, the pairing is well-defined and continuous. It is clearly  $G$ -equivariant. This proves (i).

Now, we prove (ii). The continuity of  $\alpha_{\lambda, \Gamma}$  is obvious from above inequality since if  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded, and if we let

$$M_B = \sup_{f \in B} \|f\|_{1, -d_0} < \infty,$$

then we have

$$\|\alpha_{\lambda, \Gamma}(h)\|_B = \sup_{f \in B} |\alpha_{\lambda, \Gamma}(h)(f)| \leq M_B \cdot \left( \int_{\Gamma \backslash G} \frac{1}{\|g\|_{\Gamma \backslash G}^{-d+d_0}} dg \right) \kappa(h), \quad h \in \mathcal{H}^\infty.$$

Next, the first paragraph of the proof shows that the function  $g \mapsto \lambda(\pi(g)h)$  belongs to  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . This completes the proof of (ii).

The different argument is based on results of Casselman (see Lemma 3-4). Indeed, because of the Dixmier–Malliavin, each  $h \in \mathcal{H}^\infty$  can be written in the form

$$h = \sum_{i=1}^l \pi(\beta_i) h_i,$$

for some  $\beta_i \in C_c^\infty(G)$  and  $h_i \in \mathcal{H}^\infty$ . Hence, we have

$$\alpha_{\lambda, \Gamma}(h) = \sum_{i=1}^l r'(\beta_i) \alpha(h_i)$$

which implies that  $\alpha_{\lambda, \Gamma}(h) \in \mathcal{A}_{umg}(\Gamma \backslash G)$ .

Now, we prove (iii). Let  $f \in C_c^\infty(G)$ . Then,  $P_\Gamma(f)(x) \stackrel{\text{def}}{=} \sum_{\gamma \in \Gamma} f(\gamma x)$  for  $x \in G$ , defines an element of  $\mathcal{S}(\Gamma \backslash G)$  which is compactly supported modulo  $\Gamma$ . For  $h \in \mathcal{H}^\infty$ , we have

$$\alpha_{\lambda, \Gamma}(h) (P_\Gamma(f)) = \int_{\Gamma \backslash G} \lambda(\pi(g)h) P_\Gamma(f)(g) dg = \int_G \lambda(\pi(g)h) f(g) dg.$$



Letting  $f \in C_c^\infty(G)$  vary, we see that there exists at least one  $h \in \mathcal{H}^\infty$  such that  $\alpha_{\lambda, \Gamma}(h) \neq 0$  provided that  $\lambda \neq 0$ . In view of Lemma 4-3, this implies that  $\alpha_{\lambda, \Gamma}$  is an embedding. Next, as in the proof of Lemma 4-3, we define projectors

$$E_\delta(\alpha) = \int_K d(\delta) \overline{\xi_\delta(k)} r'(k) \alpha \, dk, \quad \alpha \in \mathcal{S}(\Gamma \backslash G)'$$

for  $\delta \in \hat{K}$ . Since  $\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)$  is obviously dense in  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$ , we have that

$$E_\delta(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$$

is dense in

$$E_\delta(Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))).$$

But

$$E_\delta(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)) = \alpha_{\lambda, \Gamma}(E_\delta(\mathcal{H}^\infty)) = \alpha_{\lambda, \Gamma}(\mathcal{H}^\infty(\delta)) = \alpha_{\lambda, \Gamma}(\mathcal{H}_K(\delta))$$

is a finite-dimensional space. Hence, it is closed. Thus, we have that

$$Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))(\delta) \stackrel{def}{=} E_\delta(Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))) = \alpha_{\lambda, \Gamma}(\mathcal{H}_K(\delta))$$

is finite-dimensional. This proves that  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is admissible. We show that  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is irreducible i.e., only closed  $G$ -invariant subspaces of  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  are  $\{0\}$  and  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$ . We use smooth vectors.

Using the argument from ([24], Lemma 1.6.4), the subspace of smooth vectors  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty$  in  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is a complete locally convex representation of  $G$  where topology is defined by the semi-norms:

$$\alpha \mapsto \|r'(u)\alpha\|_B,$$

where  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  and  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded. The key thing is that each smooth vector has a Fourier expansion analogous to the one in the proof of Lemma 4-3. Then, as in the proof of Lemma 4-3 we see that  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty$  is irreducible meaning that only  $G$ -invariant subspaces are trivial and everything.

Now, if  $W \subset Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is closed  $G$ -invariant subspace. Assume  $W \neq 0$ . Then

$$W^\infty \subset Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty$$

is closed  $G$ -invariant subspace in appropriate topology. It is dense in  $W$  (see [13], Corollary 1), and therefore non-zero. But then we must have

$$W^\infty = Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))^\infty.$$

Again because the smooth vectors are dense ([13], Corollary 1), this implies

$$W = Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$$

□

The Garding space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  has a natural filtration by the smooth Frechét representations:

$$\mathcal{S}(\Gamma \backslash G) \subset \cdots \subset \mathcal{A}_{umg, -1}(\Gamma \backslash G) \subset \mathcal{A}_{umg, 0}(\Gamma \backslash G) \subset \mathcal{A}_{umg, 1}(\Gamma \backslash G) \subset \mathcal{A}_{umg, 2}(\Gamma \backslash G) \subset \cdots,$$

where for an integer  $n$  we let

$$\mathcal{A}_{umg,n}(\Gamma \backslash G) = \{\varphi \in \mathcal{A}_{umg}(\Gamma \backslash G); \quad \|\varphi\|_{u,n} < \infty, \quad u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\}.$$

We remark that all embeddings are continuous, and that

$$\mathcal{S}(\Gamma \backslash G) = \bigcap_{n \in \mathbb{Z}} \mathcal{A}_{umg,n}(\Gamma \backslash G)$$

We may therefore let

$$\mathcal{A}_{umg,-\infty}(\Gamma \backslash G) = \mathcal{S}(\Gamma \backslash G).$$

**Lemma 4-6.** *Let  $n \geq -\infty$ . Then, the representation of  $G$  on  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  is of moderate growth.*

*Proof.* The proof is similar to the proof of ([25], Lemma 11.5.1). We remarked above that representation is smooth. Let  $\rho$  be the continuous seminorm on  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ . Then, since  $\rho$  is continuous, there exists a constant  $C > 0$ , and  $u_1, \dots, u_l \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  such that

$$\|\varphi\|_{\rho} \leq C \cdot (\|\varphi\|_{u_1,n} + \dots + \|\varphi\|_{u_l,n}), \quad \varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G).$$

This is for the case  $n > -\infty$ . But when  $n = -\infty$ , the above is true for convenient integer (again denoted by)  $n$ . In this case, we fix such  $n$ .

Next, we consider the standard filtration of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  by finite  $G$ -invariant subspaces:

$$\mathcal{U}^0(\mathfrak{g}_{\mathbb{C}}) = \mathbb{C} \subset \mathcal{U}^1(\mathfrak{g}_{\mathbb{C}}) \subset \mathcal{U}^2(\mathfrak{g}_{\mathbb{C}}) \subset \dots$$

Let  $k \geq 0$ . Let  $v_1, \dots, v_k$  be the basis of  $\mathcal{U}^k(\mathfrak{g}_{\mathbb{C}})$ . Then, there exists smooth functions  $\eta_{i,j}$  such that

$$Ad(g)v_i = \sum_{j=1}^k \eta_{ij}(g)v_j.$$

Clearly,  $\eta_{i,j}$  are matrix coefficients of the representation on  $\mathcal{U}^k(\mathfrak{g}_{\mathbb{C}})$ . By the construction of the norm, there exists  $D, r > 0$  such that

$$|\eta_{ij}(g)| \leq D \cdot \|g\|^r, \quad g \in G,$$

for all  $i, j$ .

We assume that  $k$  is large enough so that  $u_1, \dots, u_l \in \mathcal{U}^k(\mathfrak{g}_{\mathbb{C}})$ . Then, we can write

$$Ad(g)u_i = \sum_{j=1}^k \nu_{ij}(g)v_j.$$

The functions  $\nu_{ij}$  are linear combinations of functions  $\eta_{ij}$ . Therefore, there exists  $D_1 > 0$  such that

$$|\nu_{ij}(g)| \leq D_1 \cdot \|g\|^r, \quad g \in G,$$

for all  $i, j$ .

Now, for  $\varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G)$ , and  $g \in G$ , using properties of the norm, we have

$$\begin{aligned}
 \|r(g)\varphi\|_\rho &\leq C \cdot \sum_{i=1}^l \|\varphi\|_{u_i,n} = C \cdot \sum_{i=1}^l \sup_{x \in G} \|x\|^{-n} |u_i \cdot r(g)\varphi(x)| \\
 &= C \cdot \sum_{i=1}^l \sup_{x \in G} \|x\|^{-n} |r(g)Ad(g^{-1}u_i) \cdot \varphi(x)| \\
 &= C \cdot \sum_{i=1}^l \sup_{x \in G} \|x\|^{-n} |(Ad(g^{-1})u_i) \cdot \varphi(xg)| \\
 &\leq C \cdot \sum_{i=1}^l \sum_{j=1}^k |\nu_{ij}(g^{-1})| \sup_{x \in G} \|x\|^{-n} |v_j \cdot \varphi(xg)| \\
 &= C \cdot \sum_{i=1}^l \sum_{j=1}^k |\nu_{ij}(g^{-1})| \sup_{x \in G} \|xg^{-1}\|^{-n} |v_j \cdot \varphi(x)| \\
 &\leq CD_1 \cdot \sum_{i=1}^l \sum_{j=1}^k \|g^{-1}\|^r \sup_{x \in G} \|xg^{-1}\|^{-n} |v_j \cdot \varphi(x)| \\
 &\leq CD_1 \cdot \sum_{i=1}^l \sum_{j=1}^k \|g\|^{n+r} \sup_{x \in G} \|x\|^{-n} |v_j \cdot \varphi(x)| \\
 &= lCD_1 \|g\|^{n+r} \sum_{j=1}^k \|\varphi\|
 \end{aligned}$$

□

**Lemma 4-7.** *Let  $n \geq -\infty$ . Then, the linear functional  $\varphi \mapsto \varphi(1)$  is continuous on  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ .*

*Proof.* Assume first that  $n > -\infty$ . Then, we have

$$|\varphi(1)| = \|1\|_{\Gamma \backslash G}^{-n} |\varphi(1)| \leq \|\varphi\|_{1,-n},$$

for  $\varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G)$ . The case  $n = -\infty$  is a consequence of above inequalities. □

**Lemma 4-8.** *Let  $\Gamma \subset G$  be a discrete subgroup. Let  $\lambda$  be a continuous functional on  $\mathcal{H}^\infty$  which is  $\Gamma$ -invariant. For any integer  $n$  such that  $n > d_{\pi,\lambda}$ , the map which assigns to  $h \in \mathcal{H}^\infty$  a function  $g \mapsto \lambda(\pi(g)h)$  in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  is continuous,  $G$ -equivariant, and if  $\lambda \neq 0$ , then it is an embedding. Moreover, the same holds if  $n = d_{\pi,\lambda} = -\infty$ .*

*Proof.* By definition of  $d_{\pi,\lambda}$  (see Definition 4-2) and the fact that  $\|x\| \geq 1$  for all  $x \in G$ , we have (see (4-1))

$$|\lambda(\pi(g)h)| \leq \|g\|^d \kappa(h), \quad g \in G, \quad h \in \mathcal{H}^\infty.$$

The semi-norm  $h \mapsto \kappa(\pi(u)h)$  is again continuous, for  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , and we have as a consequence of above inequality

$$|\lambda(\pi(g)\pi(u)h)| \leq \|g\|^d \kappa(\pi(u)h), \quad g \in G, \quad h \in \mathcal{H}^\infty.$$

For  $\gamma \in \Gamma$ , the  $\Gamma$ -invariance of  $\lambda$  implies that

$$|\lambda(\pi(g)\pi(u)h)| = |\lambda(\pi(\gamma g)\pi(u)h)| \leq \|\gamma g\|^d \kappa(\pi(u)h)$$

Since the norm is continuous and  $\Gamma$  discrete, for fixed  $x \in G$ , there exists  $\gamma_0 \in \Gamma$  such that

$$\|\gamma_0 x\| = \|x\|_{\Gamma \backslash G} = \inf_{\gamma \in \Gamma} \|\gamma x\|.$$

Thus above inequality implies

$$|\lambda(\pi(g)\pi(u)h)| \leq \|g\|_{\Gamma \backslash G}^d \kappa(\pi(u)h).$$

This implies that

$$\sup_{g \in G} \|g\|_{\Gamma \backslash G}^{-d} |\lambda(\pi(g)\pi(u)h)| \leq \kappa(\pi(u)h).$$

Now, the lemma easily follows.  $\square$

Now, we prove the main result of the present section.

**Theorem 4-9.** *Let  $\mathcal{V} \subset \mathcal{S}(\Gamma \backslash G)'$  be a closed irreducible admissible subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$ . Then, there exists an irreducible admissible representation of  $G$  acting on the Hilbert space  $\mathcal{H}$ , and a non-zero  $\Gamma$ -invariant continuous functional on  $\mathcal{H}^\infty$  such that*

$$\mathcal{V} = Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)).$$

*Proof.* By ([13], Lemma 4),  $\mathcal{V}^\infty \cap \mathcal{V}_K$  is dense in  $\mathcal{V}$ . Since  $\mathcal{V}$  is admissible, we see that  $\mathcal{V}_K \subset \mathcal{V}^\infty$ . It is easy to check that  $\mathcal{V}_K$  is an irreducible  $(\mathfrak{g}, K)$ -module. In particular, every vector in  $\mathcal{V}_K$  is  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite. Therefore, by ([13], Theorem 1), for each  $\varphi \in \mathcal{V}_K$  there exists  $\alpha \in C_c^\infty(G)$  such that  $r'(\alpha)\varphi = \varphi$ . Hence,  $\mathcal{V}_K$  belongs to the Garding space of  $\mathcal{V}$ , and consequently to the Garding space of  $\mathcal{S}(\Gamma \backslash G)'$  which is  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . By means of the Casselman subrepresentation theorem, we can find an infinitesimal embedding of  $\mathcal{V}_K$  into a principal series of  $G$ . In this way, we obtain a globalization of  $\mathcal{V}_K$  i.e., there exists an irreducible admissible representation  $(\pi, \mathcal{H})$  on the Hilbert space  $\mathcal{H}$  infinitesimally equivalent to  $\mathcal{V}_K$ . Let us fix an isomorphism  $\eta : \mathcal{H}_K \rightarrow \mathcal{V}_K$ .

We recall the filtration of  $\mathcal{A}_{umg}(\Gamma \backslash G)$  by the representations of moderate growth (see Lemma 4-6) :

$$\mathcal{A}_{umg,1}(\Gamma \backslash G) \subset \mathcal{A}_{umg,2}(\Gamma \backslash G) \subset \mathcal{A}_{umg,3}(\Gamma \backslash G) \subset \cdots.$$

This is also filtration of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ -modules. Since  $\mathcal{V}_K$  is irreducible, there exists  $n \geq 1$  such that

$$\mathcal{V}_K \subset \mathcal{A}_{umg,n}(\Gamma \backslash G).$$

Let  $V_n$  be the closure of  $\mathcal{V}_K$  in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ . It is obvious that a  $(\mathfrak{g}, K)$ -module on the space of  $K$ -finite vectors in  $W_n$  is  $\mathcal{V}_K$ . Therefore,  $V_n$  is irreducible. We remark that  $V_n$  being a closed subrepresentation of a representation of moderate growth  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  is also a representation of moderate growth (see Lemma 4-6; [25], Lemma 11.5.2). But  $\mathcal{H}^\infty$  is also a representation of moderate growth ([25], Lemma 11.5.1) and irreducible (see Lemma 4-3).

So, the isomorphism  $\eta : \mathcal{H}_K \longrightarrow (W_n)_K = \mathcal{V}_K$ , extends to a continuous isomorphism of  $G$ -representations  $\eta : \mathcal{H}^\infty \longrightarrow V_n$  applying ([25], Theorem 11.5.1).

Now, the required linear functional is

$$\lambda(h) \stackrel{\text{def}}{=} \eta(h)(1).$$

Indeed, it is obviously continuous (see Lemma 4-7). Next, it is  $\Gamma$ -invariant since

$$\lambda(\pi(\gamma)h) = \eta(\pi(\gamma)h)(1) = r(\gamma)\eta(h)(1) = \eta(h)(\gamma) = \eta(h)(1), \quad h \in \mathcal{H}^\infty, \quad \gamma \in \Gamma.$$

Now, using the notation introduced in Proposition 4-4, we compute

$$\lambda(\pi(g)h) = \eta(\pi(g)h)(1) = r(g)\eta(h)(1) = \eta(h)(g), \quad h \in \mathcal{H}^\infty, \quad g \in G.$$

For  $h \in \mathcal{H}_K$ , above computation and Proposition 4-4 (ii) implies that

$$\alpha_{\lambda, \Gamma}(\mathcal{H}_K) = \mathcal{V}_K.$$

The proof of Proposition 4-4 shows that the space of  $K$ -finite vectors of  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$  is  $\alpha_{\lambda, \Gamma}(\mathcal{H}_K)$  and it is dense in  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$ . Since  $\mathcal{V}_K$  is dense in  $\mathcal{V}_K$ , the theorem follows.  $\square$

Examples of subrepresentation can explicitly be constructed using Eisenstein series [14], or be shown to exists using Poincaré series [21], or the trace formula ([1], [2]). Now, we show that there are no finite-dimensional representations except the trivial representation in  $\mathcal{S}(\Gamma \backslash G)'$  under appropriate assumptions.

**Theorem 4-10.** *We maintain Assumption 2-2. Then, if  $G$  has no compact components, then the trivial representation is the only finite-dimensional subrepresentation of  $\mathcal{S}(\Gamma \backslash G)'$ .*

*Proof.* Let  $\mathcal{V} \subset \mathcal{S}(\Gamma \backslash G)'$  be a finite-dimensional subrepresentation. Then, by Theorem 4-9, there exists finite dimensional representation  $\mathcal{H}$  (satisfying the assumption of the first paragraph of this section) such that

$$\mathcal{V} = Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty)) = \alpha_{\lambda, \Gamma}(\mathcal{H}^\infty).$$

Since  $\mathcal{H}$  is finite-dimensional, we have  $\mathcal{H}^\infty = \mathcal{H}$ , and there is a non-zero  $\Gamma$ -invariant functional on  $\mathcal{H}$ . So, the algebraic dual  $\mathcal{H}'$  is smooth irreducible representation of  $G$  having a non-zero  $\Gamma$ -invariant vector. By general theory,  $\mathcal{H}'$  is a restriction of an algebraic (holomorphic) representation of  $\mathcal{G}(\mathbb{C})$  to  $G$ . But the Borel density theorem [3] implies that  $\Gamma$  is Zariski dense in  $\mathcal{G}(\mathbb{C})$ . Because of that a  $\Gamma$ -invariant vector is also  $\mathcal{G}(\mathbb{C})$ -invariant. In particular, it is  $G$ -invariant. But  $\mathcal{H}'$  is an irreducible representation of  $G$ . Hence,  $\mathcal{H}'$  is one-dimensional and  $G$  acts trivially. Thus, the same holds for  $\mathcal{H}$  and consequently for  $\mathcal{V}$ .  $\square$

The most important consequence of Theorem 4-10 is the following corollary:

**Corollary 4-11.** *We maintain Assumption 2-2. Then, if  $G$  has no compact components, then the trivial representation is the only finite-dimensional subrepresentation of a  $(\mathfrak{g}, K)$ -module  $\mathcal{A}(\Gamma \backslash G)$  (defined in Section 2).*

5. RESULTS ON  $L^2(\Gamma \backslash G)$ 

In this section we continue with the assumptions of previous Section 4. The reader should review the second paragraph of Section 4.

We consider the usual embedding  $\mathcal{A}_{umg}(\Gamma \backslash G) \hookrightarrow \mathcal{S}(\Gamma \backslash G)'$ , given by  $\varphi \mapsto \beta_\varphi$  where  $\beta_\varphi$  is defined by  $\beta_\varphi(f) = \int_{\Gamma \backslash G} \varphi(x) f(x) dx$ , for  $f \in \mathcal{S}(\Gamma \backslash G)$ .

**Lemma 5-1.** *We equip the space of smooth vectors  $(\mathcal{S}(\Gamma \backslash G)')^\infty$  with the usual topology (described in the proof below). Let  $n \geq -\infty$ . Then, the embedding  $\mathcal{A}_{umg,n}(\Gamma \backslash G) \hookrightarrow (\mathcal{S}(\Gamma \backslash G)')^\infty$ , given by  $\varphi \mapsto \beta_\varphi$ , is  $G$ -equivariant and continuous.*

*Proof.* Recall that the space of smooth vectors  $(\mathcal{S}(\Gamma \backslash G)')^\infty$  in  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex representation of  $G$  where topology is defined by the semi-norms:

$$\alpha \mapsto \|r'(u)\alpha\|_B,$$

where  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  and  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded.

Let  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$  and let  $B \subset \mathcal{S}(\Gamma \backslash G)$  be a bounded set. Let  $\varphi \in \mathcal{A}_{umg,n}(\Gamma \backslash G)$ . Then, assuming that in the computation below  $n$  means any integer if originally we have  $n = -\infty$ ,

$$\sup_{f \in B} |r'(u)\beta_\varphi(f)| = \sup_{f \in B} \left| \int_{\Gamma \backslash G} u \cdot \varphi(x) f(x) dx \right|$$

We note again ([9], Proposition 1.9) implies that  $\int_{\Gamma \backslash G} \|g\|_{\Gamma \backslash G}^{-d_0} dg < \infty$  for all sufficiently large  $d_0 > 0$ . Let  $M_B = \sup_{f \in B} \|f\|_{1, -d_0} < \infty$ . Hence

$$\sup_{f \in B} |r'(u)\beta_\varphi(f)| \leq \left( M_B \cdot \int_{\Gamma \backslash G} \|g\|_{\Gamma \backslash G}^{n-d_0} dg \right) \|u \cdot \varphi\|_{u,n}.$$

The continuity of the map easily follows. The map is obviously  $G$ -equivariant.  $\square$

We recall the classical and well-known argument in our settings. Let  $X \in \mathfrak{g}$ . Then, for  $F \in L^1(\Gamma \backslash G) \cap C^\infty(G)$  and  $f \in \mathcal{S}(\Gamma \backslash G)$ , we have

$$\begin{aligned} & \int_{\Gamma \backslash G} X \cdot F(x) f(x) dx = \\ &= \int_{\Gamma \backslash G} \frac{d}{dt} \Big|_{t=0} F(x \exp(tX)) f(x) dx \\ &= \int_{\Gamma \backslash G} \frac{d}{dt} \Big|_{t=0} (F(x \exp(tX)) f(x \exp(tX))) dx - \int_{\Gamma \backslash G} F(x) \frac{d}{dt} \Big|_{t=0} f(x \exp(tX)) dx \\ &= \int_{\Gamma \backslash G} \frac{d}{dt} \Big|_{t=0} (F(x) f(x)) dx - \int_{\Gamma \backslash G} F(x) \frac{d}{dt} \Big|_{t=0} f(x \exp(tX)) dx \\ &= - \int_{\Gamma \backslash G} F(x) \frac{d}{dt} \Big|_{t=0} f(x \exp(tX)) dx. \end{aligned}$$

The map  $\mathfrak{g} \rightarrow \mathfrak{g}$ , given by  $X \mapsto -X$ . This extends to a  $\mathbb{C}$ -linear anti-automorphism  $u \mapsto u^\#$  of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  which satisfies

$$(5-2) \quad \int_{\Gamma \backslash G} u.F(x) f(x) dx = \int_{\Gamma \backslash G} F(x) u^\#.f(x) dx$$

Since  $\mathcal{S}(\Gamma \backslash G)$  is a smooth representation, for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ , the map  $f \mapsto u.f$  is continuous. So, if  $\beta \in \mathcal{S}(\Gamma \backslash G)'$ , then  $f \mapsto \beta(u.f)$  is a continuous linear functional. Hence,  $\mathcal{S}(\Gamma \backslash G)'$  becomes  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ -module:

$$u.\beta(f) = \beta(u^\#.f), \quad f \in \mathcal{S}(\Gamma \backslash G).$$

We consider the embedding of  $L^2(\Gamma \backslash G) \hookrightarrow \mathcal{S}(\Gamma \backslash G)'$ , given by  $\varphi \mapsto \beta_\varphi$  where  $\beta_\varphi$  is defined by  $\beta_\varphi(f) = \int_{\Gamma \backslash G} \varphi(x) f(x) dx$ , for  $f \in \mathcal{S}(\Gamma \backslash G)$ . It is proved in ([9], Proposition 1.17) that the map is continuous. We sketch the argument. Let  $d > 0$  be an integer such that  $\int_{\Gamma \backslash G} \|x\|_{\Gamma \backslash G}^{-2d} dx < \infty$ . Let  $B \subset \mathcal{S}(\Gamma \backslash G)$  be a bounded set. Then, we have the following:

$$(5-3) \quad \begin{aligned} \|\beta_\varphi\|_B &= \sup_{f \in B} \left| \int_{\Gamma \backslash G} \varphi(x) f(x) dx \right| \leq \sup_{f \in B} \int_{\Gamma \backslash G} |\varphi(x)| |f(x)| dx \\ &= \left( \int_{\Gamma \backslash G} \|x\|_{\Gamma \backslash G}^{-2d} dx \right)^{1/2} \left( \sup_{f \in B} \|f\|_{1,-d} \right) \cdot \left( \int_{\Gamma \backslash G} |\varphi(x)|^2 dx \right)^{1/2} \end{aligned}$$

which clearly proves the continuity. It is by the general theory that we have continuous map of smooth representations  $(L^2(\Gamma \backslash G))^\infty \hookrightarrow (\mathcal{S}(\Gamma \backslash G)')^\infty$ , the image is actually in  $\mathcal{A}_{umg}(\Gamma \backslash G)$  (see Lemma 3-5). But even more is true

**Lemma 5-4.** *If the sequence  $(\varphi_n)_{n \geq 1}$  in  $(L^2(\Gamma \backslash G))^\infty$  converges to  $\varphi \in L^2(\Gamma \backslash G)$ , then for each  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  the sequence  $(\beta_{r(u)\varphi_n})_{n \geq 1}$  converges to  $u.\beta_\varphi$  in the topology of  $\mathcal{S}(\Gamma \backslash G)'$ .*

*Proof.* Arguing as in (5-3) and using (5-2), we have

$$\|\beta_{u.\varphi_n} - u.\beta_\varphi\|_B \leq \left( \int_{\Gamma \backslash G} \|x\|_{\Gamma \backslash G}^{-2d} dx \right)^{1/2} \left( \sup_{f \in B} \|f\|_{u^\#,-d} \right) \cdot \left( \int_{\Gamma \backslash G} |\varphi_n(x) - \varphi(x)|^2 dx \right)^{1/2},$$

for all bounded sets  $B \subset \mathcal{S}(\Gamma \backslash G)$  and  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ .  $\square$

**Corollary 5-5.**  $\beta_\varphi$  is a smooth vector in  $\mathcal{S}(\Gamma \backslash G)'$  for all  $\varphi \in L^2(\Gamma \backslash G)$ .

*Proof.* We recall that the space of smooth vectors  $(\mathcal{S}(\Gamma \backslash G)')^\infty$  in  $\mathcal{S}(\Gamma \backslash G)'$  is a complete locally convex representation of  $G$  where topology is defined by the semi-norms:

$$\alpha \mapsto \|r'(u)\alpha\|_B,$$

where  $u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  and  $B \subset \mathcal{S}(\Gamma \backslash G)$  is bounded. Now, since by the general theory, the image of  $(L^2(\Gamma \backslash G))^\infty$  belongs to  $(\mathcal{S}(\Gamma \backslash G)')^\infty$ , we may apply Lemma 5-4 to complete the proof.  $\square$

**Lemma 5-6.** *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Then, we have the following commutative diagram:*

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\subset} & L^2(\Gamma \backslash G) & \xrightarrow{\varphi \mapsto \beta_\varphi} & \mathcal{S}(\Gamma \backslash G)' \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{H}^\infty & \xrightarrow{\subset} & (L^2(\Gamma \backslash G))^\infty & \xrightarrow{\varphi \mapsto \beta_\varphi} & \mathcal{A}_{umg}(\Gamma \backslash G), \end{array}$$

where in the first row are continuous maps, and the second row is also continuous if we equip  $\mathcal{A}_{umg}(\Gamma \backslash G)$  with the topology inherited from  $(\mathcal{S}(\Gamma \backslash G)')^\infty$ .

*Proof.* Above discussions imply that the first row consists of continuous maps. Next, Lemma 3-4 and Dixmier–Malliavin theorem [12] assure that the image of  $(L^2(\Gamma \backslash G))^\infty$  in  $\mathcal{A}_{umg}(\Gamma \backslash G)$ . Finally, the commutativity of the diagram is a consequence of general facts about smooth vectors.  $\square$

The following result uses deep results about globalization due to Casselman [8] and Wallach [25].

**Lemma 5-7.** *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Then, there exists  $n_0 \in \mathbb{Z}$  such that for  $n \geq n_0$ , the map  $\varphi \mapsto \beta_\varphi$  maps  $\mathcal{H}^\infty$  equipped with its natural topology into  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  (considered as a subspace of  $\mathcal{S}(\Gamma \backslash G)'$  but equipped with its standard topology) isomorphically onto its image which is closed in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$ .*

*Proof.* By Lemma 5-6, the map  $\mathcal{H}^\infty \rightarrow \mathcal{A}_{umg}(\Gamma \backslash G)$ , given by  $\varphi \mapsto \beta_\varphi$ , is continuous if we equip  $\mathcal{A}_{umg}(\Gamma \backslash G)$  with the topology inherited from  $(\mathcal{S}(\Gamma \backslash G)')^\infty$ . It is also  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ -equivariant. Select any non-zero  $\varphi \in \mathcal{H}_K$ . Then, there exists  $n_0 \in \mathbb{Z}$  such that

$$\varphi \in \mathcal{A}_{umg,n_0}(\Gamma \backslash G) \subset \mathcal{A}_{umg,n_0+1}(\Gamma \backslash G) \subset \mathcal{A}_{umg,n_0+2}(\Gamma \backslash G) \subset \cdots$$

But since  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  are smooth representations, and  $\mathcal{H}_K$  is an irreducible  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ -module, we see that the image of  $\mathcal{H}_K$  is contained in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  for  $n \geq n_0$ . Let  $W_n$  be the closure of the image in  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  for  $n \geq n_0$ . It is obvious that a  $(\mathfrak{g}, K)$ -module on the space of  $K$ -finite vectors in  $W_n$  is the image of  $\mathcal{H}_K$ . Therefore,  $W_n$  is irreducible. We remark that  $W_n$  being a closed subrepresentation of a representation of moderate growth  $\mathcal{A}_{umg,n}(\Gamma \backslash G)$  is also a representation of moderate growth (see Lemma 4-6; [25], Lemma 11.5.2). But  $\mathcal{H}^\infty$  is also a representation of moderate growth ([25], Lemma 11.5.1) and irreducible (see Lemma 4-3). So, the map  $\mathcal{H}_K \rightarrow (W_n)_K$ ,  $\varphi \mapsto \beta_\varphi$ , which is an isomorphism of  $(\mathfrak{g}, K)$ -modules, extends to a continuous isomorphism of  $G$ -representations  $\mathcal{H}^\infty$  and  $W_n$  applying ([25], Theorem 11.5.1). Let us finally determine this map. This is easy since by the composition with the continuous inclusion  $\mathcal{A}_{umg,n}(\Gamma \backslash G) \hookrightarrow (\mathcal{S}(\Gamma \backslash G)')^\infty$  (see Lemma 5-1), we obtained the map that coincides on  $\mathcal{H}_K$  with the continuous map given by the second row of the diagram in Lemma 5-6. Hence, the map is  $\varphi \mapsto \beta_\varphi$ .  $\square$

The first main result of this section is the following theorem. The reader should review the statement of Proposition 4-4.



**Theorem 5-8.** *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Then, we have the following:*

- (i) *The continuous inclusion  $\mathcal{H} \hookrightarrow L^2(\Gamma \backslash G)$  gives rise to a continuous linear functional  $\lambda$  such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\varphi \mapsto \beta \varphi} & \mathcal{S}(\Gamma \backslash G)' \\ \subset \uparrow & & = \uparrow \\ \mathcal{H}^\infty & \xrightarrow{\alpha_{\lambda, \Gamma}} & \mathcal{S}(\Gamma \backslash G)'. \end{array}$$

*Furthermore,  $\mathcal{H}$  is embedded into the smooth vectors of the closure  $Cl(\alpha_{\lambda, \Gamma}(\mathcal{H}^\infty))$ .*

- (ii) *In addition, assume that Assumption 2-2 holds. Then, if  $\mathcal{H}$  is tempered, then  $d_{\mathcal{H}, \lambda} = -\infty$ .*

*Proof.* We prove (i). Lemma 5-6 implies that  $\mathcal{H}^\infty \subset \mathcal{A}_{umg}(\Gamma \backslash G)$ . Next, by Lemma 5-7, there exists an integer  $n \geq 1$  such that  $\mathcal{H}^\infty \subset \mathcal{A}_{umg, n}(\Gamma \backslash G)$ . This inclusion is continuous in appropriate topologies. Hence, by Lemma 4-7,  $\varphi \mapsto \varphi(1)$  is a continuous functional on  $\mathcal{H}^\infty$ . If we denote this functional by  $\lambda$ , then the commutativity of the diagram follows. The last part of (i) follows from Corollary 5-5.

We prove (ii). Because of the Assumption 2-2, we may consider the space of the closed subspace  $L_{cusp}^2(\Gamma \backslash G)$  of cuspidal functions in  $L^2(\Gamma \backslash G)$ . It is a  $G$ -subrepresentation. By a result of Wallach [23], since  $\mathcal{H}$  is a tempered closed subrepresentation of  $L^2(\Gamma \backslash G)$ ,  $\mathcal{H}$  is a closed subrepresentation of  $L_{cusp}^2(\Gamma \backslash G)$ . Then, using the notation of Section 2,  $\mathcal{H}_K \subset \mathcal{A}_{cusp}(\Gamma \backslash G)$ , and in fact

$$\mathcal{H}_K \subset \mathcal{A}_{cusp}(\Gamma \backslash G) \cap \mathcal{S}(\Gamma \backslash G),$$

since  $K$ -finite cuspidal automorphic forms are rapidly decreasing [7]. Now, arguing as in the proof of Lemma 5-7, we see that

$$\mathcal{H}^\infty \subset \mathcal{A}_{cusp}^\infty(\Gamma \backslash G) \cap \mathcal{S}(\Gamma \backslash G).$$

This implies (ii). It also shows that smooth cuspidal automorphic forms are rapidly decreasing. Which gives a different proof of the fact proved also in [17].  $\square$

We maintain the Assumption 2-2, and assume that  $G$  posses representations in discrete series ([16], [13]). Then, if  $(\pi, \mathcal{H})$  is a representation in discrete series, then there exists infinitely many congruence subgroups  $\Gamma$  such that we can embed it in  $L_{cusp}^2(\Gamma \backslash G)$  ([22], [10]). Therefore, it posses a non-zero  $\Gamma$  invariant functional such that  $d_{\pi, \lambda} = -\infty$ . On the other hand, by counting tempered representation, most of them do not appear as subrepresentations of  $L^2(\Gamma \backslash G)$  for a congruence subgroup  $\Gamma$ .

Following Harish-Chandra ([13], Section 5), we introduce the topology on  $C^\infty(G)$  by means of seminorms

$$\nu_{\Omega, u}, \quad \Omega \subset G \text{ is compact and } u \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$$

defined by

$$\nu_{\Omega, u}(F) = \sup_{x \in \Omega} \|uF(x)\|.$$

We end this section with the following theorem:

**Theorem 5-9.** *Let  $\mathcal{H}$  be a closed irreducible  $G$ -invariant subspace of  $L^2(\Gamma \backslash G)$ . Assume that the sequence of elements in  $\mathcal{H}^\infty$ ,  $(\varphi_n)_{n \geq 1}$ , converges to  $\varphi \in \mathcal{H}^\infty$  in the standard topology on  $\mathcal{H}^\infty$ . Then, it converges to  $\varphi$  in above described topology on  $C^\infty(G)$ . In addition, assume that Assumption 2-2 holds. Then, if  $\mathcal{H} \subset L^2_{\text{cusp}}(\Gamma \backslash G)$ , then  $u.\varphi_n \mapsto u.\varphi$  uniformly on  $G$  for all  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ .*

*Proof.* The first part follows from Lemma 5-7. In addition, for the second part, we need the fact that

$$\mathcal{H}^\infty \subset \mathcal{A}_{\text{cusp}}^\infty(\Gamma \backslash G) \cap \mathcal{S}(\Gamma \backslash G)$$

established in the proof of Theorem 4-11. □

## 6. ON $\Gamma$ -INVARIANTS IN $\mathcal{S}'(G)$

Let  $\Gamma \subset G$  be a discrete subgroup. Then, the canonical map  $\mathcal{S}(G) \rightarrow \mathcal{S}(\Gamma \backslash G)$ , given by  $P_\Gamma(f)(x) = \sum_{\gamma \in \Gamma} f(\gamma x)$ , is a continuous ([9], Proposition 1.110). We sketch the argument since the details of the argument will be useful later. Let  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ . Let  $n \geq 1$  be an integer. Then, we have

$$\|P_\Gamma(f)\|_{u, -n} = \sup_{x \in G} \|x\|_{\Gamma \backslash G}^n |u.P_\Gamma(f)(x)|.$$

Since  $u.P_\Gamma(f) = P_\Gamma(u.f)$  and  $\|x\|_{\Gamma \backslash G} \leq \|\gamma x\|$ , we obtain

$$\begin{aligned} \|x\|_{\Gamma \backslash G}^n |u.P_\Gamma(f)(x)| &= \|x\|_{\Gamma \backslash G}^n |P_\Gamma(u.f)(x)| \\ &\leq \sum_{\gamma \in \Gamma} \|\gamma x\|^n \cdot |u.f(\gamma x)| \leq \|f\|_{u, -d-n} \left( \sum_{\gamma \in \Gamma} \|\gamma x\|^{-d} \right), \end{aligned}$$

where  $d > 0$  is large enough such that  $\int_G \|x\|^{-d} dx < \infty$ . But, by ([9], Lemma 1.10), we have

$$(6-1) \quad M_d \stackrel{\text{def}}{=} \sup_{x \in G} \sum_{\gamma \in \Gamma} \|\gamma x\|^{-d} < \infty.$$

Thus, we obtain

$$(6-2) \quad \|P_\Gamma(f)\|_{u, -n} \leq M_d \|f\|_{u, -d-n}.$$

The group  $\Gamma$  acts on the left on  $\mathcal{S}(G)$ :  $l(\gamma)f(x) = f(\gamma^{-1}x)$ . By duality  $\Gamma$  acts on  $\mathcal{S}(G)'$ :  $l'(\gamma)\alpha(f) = \alpha(l(\gamma^{-1})f)$ .

**Lemma 6-3.** *Let  $\gamma \in \Gamma$ . Then, the linear operator  $l(\gamma)$  (resp.,  $l'(\gamma)$ ) is continuous in the topology on  $\mathcal{S}(G)$  (resp.,  $\mathcal{S}(G)'$ ).*

*Proof.* Indeed, for  $u \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ , and for an integer  $n \geq 1$ , we have the following:

$$\|l(\gamma)f\|_{u, -n} = \sup_{x \in G} \|x\|^n |u.f(\gamma^{-1}x)| = \sup_{x \in G} \|\gamma x\|^n |u.f(x)| \leq \|\gamma\|^n \|f\|_{u, -n}.$$

This proves that  $l(\gamma)$  is continuous. Next, we have

$$\|l'(\gamma)\alpha\|_B = \sup_{f \in B} |\alpha(l(\gamma^{-1})f)| = \sup_{f \in l(\gamma^{-1})B} |\alpha(f)| = \|\alpha\|_{l(\gamma^{-1})B}.$$

We remark that since  $l(\gamma^{-1})$  is continuous, the set  $l(\gamma^{-1})B$  is bounded. This proves that  $l'(\gamma)$  is also continuous.  $\square$

**Lemma 6-4.** *Let  $(\mathcal{S}(G)')^\Gamma$  be the space of all  $\alpha \in \mathcal{S}(G)'$  such that  $l'(\gamma)\alpha = \alpha$  for all  $\gamma \in \Gamma$ . Then,  $(\mathcal{S}(G)')^\Gamma$  is a closed subrepresentation of  $\mathcal{S}(G)$  (where  $G$  acts by right translations).*

*Proof.* Indeed, if  $(\alpha_\lambda)_{\lambda \in \Lambda}$  is a net in  $(\mathcal{S}(G)')^\Gamma$  which converges to  $\alpha \in \mathcal{S}(G)'$  i.e., the nets  $\|\alpha_\lambda - \alpha\|_B$ , where  $B \subset \mathcal{S}(G)$  is bounded, converge to zero. Then, since for  $\gamma \in \Gamma$ , the operator  $l'(\gamma)$  is continuous, we have that the net  $l'(\gamma)\alpha_\lambda$  converges to  $l'(\gamma)\alpha$ . This implies  $l'(\gamma)\alpha = \alpha$ . Hence,  $\alpha \in (\mathcal{S}(G)')^\Gamma$ .  $\square$

**Proposition 6-5.** *We maintain Assumption 2-2. Then, the canonical map  $\mathcal{S}(\Gamma \backslash G)' \rightarrow \mathcal{S}(G)'$  is a continuous embedding with the image dense in the closed subrepresentation  $(\mathcal{S}(G)')^\Gamma$ . The space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  gets identified with the Garding space of the subrepresentation  $(\mathcal{S}(G)')^\Gamma$ .*

*Proof.* Since Assumption 2-2 holds, the canonical map  $\mathcal{S}(G) \rightarrow \mathcal{S}(\Gamma \backslash G)$ , given by  $P_\Gamma(f)(x) = \sum_{\gamma \in \Gamma} f(\gamma x)$ , is a continuous epimorphism ([9], Proposition 1.11, Theorem 2.2).

Next, the map  $\mathcal{S}(\Gamma \backslash G)' \rightarrow \mathcal{S}(G)'$  is an embedding. It is also obvious that its image is contained in  $(\mathcal{S}(G)')^\Gamma$ . Let us that it is continuous. Let  $B \subset \mathcal{S}(G)$  be a bounded set. Then, since  $P_\Gamma$  is continuous,  $P_\Gamma(B) \subset \mathcal{S}(\Gamma \backslash G)$  is a bounded set. Then, we have

$$\|\alpha \circ P_\Gamma\|_B = \sup_{f \in B} |\alpha(P_\Gamma(f))| = \|\alpha\|_{P_\Gamma(B)}.$$

This proves the continuity of the map.

The space  $\mathcal{A}_{umg}(\Gamma \backslash G)$  is the Garding space of  $\mathcal{S}(\Gamma \backslash G)'$ . Thus its image is contained in the Garding space of the subrepresentation  $(\mathcal{S}(G)')^\Gamma$ . But the Garding space of  $(\mathcal{S}(G)')^\Gamma$  is contained in the Garding space of  $\mathcal{S}(G)'$ . This space is  $\mathcal{A}_{umg}(G)$  (see Lemma 3-4). So let  $\alpha$  belong to the Garding space of  $(\mathcal{S}(G)')^\Gamma$ . Then, by what we have just said,  $\alpha$  is represented by a function  $\varphi \in \mathcal{A}_{umg}(G)$ :

$$\alpha(f) = \int_G \varphi(x)f(x)dx, \quad f \in \mathcal{S}(G).$$

Since  $\alpha$  is  $\Gamma$ -invariant, we have that  $\varphi(\gamma x) = \varphi(x)$ ,  $\gamma \in \Gamma$ ,  $x \in G$ . Now,  $\varphi \in \mathcal{A}_{umg}(\Gamma \backslash G)$ .

Finally, since  $\mathcal{A}_{umg}(\Gamma \backslash G)$  maps onto the Garding space of  $(\mathcal{S}(G)')^\Gamma$ , the space  $\mathcal{S}(\Gamma \backslash G)' \rightarrow \mathcal{S}(G)'$  maps onto a dense subspace of  $(\mathcal{S}(G)')^\Gamma$ .  $\square$

In the following proposition we give the most general construction of classical Poincaré series. In part, it generalizes ([21], Theorem 6.4).

**Proposition 6-6.** *Assume that  $\Gamma \subset G$  is a discrete subgroup. Let  $\varphi \in L^1(G)$ . Then the series  $\sum_{\gamma \in \Gamma} l(\gamma)\varphi$  converges absolutely in  $\mathcal{S}(G)'$  to an element of  $(\mathcal{S}(G)')^\Gamma$  (which is in the image of  $\mathcal{S}(\Gamma \backslash G)'$ ). Moreover, if  $\varphi$  is a smooth vector in the Banach representation  $L^1(G)$  under right-translations, then  $\sum_{\gamma \in \Gamma} l(\gamma)\varphi \in \mathcal{A}_{umg}(\Gamma \backslash G)$ .*

*Proof.* Let  $B \subset \mathcal{S}(G)$  be a bounded set. We need to show that

$$\sum_{\gamma \in \Gamma} \|l(\gamma)\varphi\|_B < \infty.$$

Since  $\mathcal{S}(\Gamma \backslash G)'$  is complete, this proves the absolute convergence.

By definition, we have

$$\begin{aligned} \|l(\gamma)\varphi\|_B &= \sup_{f \in B} \left| \int_G \varphi(\gamma^{-1}x) f(x) dx \right| \leq \sup_{f \in B} \int_G |\varphi(\gamma^{-1}x)| |f(x)| dx \\ &= \sup_{f \in B} \int_G |\varphi(x)| |f(\gamma x)| dx \\ &\leq \left( \sup_{f \in B} \|f\|_{1,-d} \right) \cdot \int_G |\varphi(x)| |\gamma x|^{-d} dx \end{aligned}$$

So, the series is

$$\leq \left( \sup_{f \in B} \|f\|_{1,-d} \right) \cdot M_d \int_G |\varphi(x)| dx < \infty,$$

where the number  $M_d$  is defined by (6-1).

The distribution in question is in fact the integration against the classical Poincaré series  $P_\Gamma(\varphi) \in L^1(\Gamma \backslash G)$ :

$$\begin{aligned} \int_G P_\Gamma(\varphi)(x) f(x) dx &= \sum_{\gamma \in \Gamma} \int_G \varphi(\gamma x) f(x) dx \\ &= \sum_{\gamma \in \Gamma} \int_G \varphi(\gamma^{-1}x) f(x) dx = \sum_{\gamma \in \Gamma} \int_G l(\gamma)\varphi(x) f(x) dx, \end{aligned}$$

for  $f \in \mathcal{S}(G)$ .

The space of smooth vectors in  $L^1(G)$ , where  $G$  acts by right translations  $r$ , is a Fréchet space under seminorms ([25], Lemma 11.5.1):

$$\|r(u)f\|_1 = \int_G |r(u)f(x)| dx, \quad u \in \mathcal{U}(\mathfrak{g}_\mathbb{C}).$$

Then, by Dixmier–Malliavin theorem [12], for smooth vector  $\varphi$  there exists, smooth vectors  $\varphi, \dots, \varphi_l$ , and  $\alpha_1, \dots, \alpha_l \in C_c^\infty(G)$  such that

$$\varphi = \sum_{i=1}^l r(\alpha_i)\varphi_i = \sum_{i=1}^l \varphi_i \star \alpha_i^\vee$$

where as usual  $\alpha_i^\vee(x) = \alpha_i(x^{-1})$ . By the standard measure-theoretic arguments, we have

$$P_\Gamma(\varphi) = \sum_{i=1}^l P_\Gamma(\varphi_i) \star \alpha_i^\vee.$$

Now, we apply Lemma 3-5. □

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